14.3/4 Tangent Slopes and Tangent Planes

Some Notation and Review

1st Partials

$$\frac{\partial z}{\partial x} = f_x(x,y) = \text{"slope in x} - \text{direction"}$$

$$\frac{\partial z}{\partial y} = f_x(x,y) = \text{"slope in y} - \text{direction"}$$

2nd Partial Derivatives

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x,y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x,y)$$

Mixed Partials:

$(f_x)_y$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x,y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x,y)$$

Entry Task:

Find all 1st and 2nd partials for

$$z = f(x,y) = x^2 - y^2$$

$\frac{\partial}{\partial x}$ $\qquad$ $\frac{\partial}{\partial y}$

$\frac{\partial z}{\partial x} = 2x$ $\qquad$ $\frac{\partial z}{\partial y} = -2y$

$\frac{\partial}{\partial x}$ $\qquad$ $\frac{\partial}{\partial y}$ $\qquad$ $\frac{\partial}{\partial x}$ $\qquad$ $\frac{\partial}{\partial y}$

$f_{xx} = 2$ $\qquad$ $f_{xy} = 0$ $\qquad$ $f_{yx} = 0$ $\qquad$ $f_{yy} = -2$

concave up?

these are always the same

concave down?

Visuals: https://www.math3d.org/9m8tLCz8

Example: Find all second partials for
$z = f(x, y) = x^4 + 3x^2y^3 + y^5$

$$f \xrightarrow{x} f_x = 4x^3 + 6xy^3 + 0 \qquad f \xrightarrow{y} f_y = 0 + 9x^2y^2 + 5y^4$$

$$f_x \xrightarrow{x} f_{xx} = 12x^2 + 6y^3$$

$$f_x \xrightarrow{y} f_{xy} = \underline{18xy^2}$$

$$f_y \xrightarrow{x} f_{yx} = \underline{18xy^2}$$

$$f_y \xrightarrow{y} f_{yy} = 18x^2y + 20y^3$$

same!

Clairaut's theorem: The mixed 2nd partials are continuous, then they are equal.

A few random HW Notes:

From 14.3(1)

5. **Question Details**

Find the first partial derivatives of the function.

$f(x, y) = x^{5y}$

$f_x(x, y) =$

$f_y(x, y) =$

$$z = x^{5y}$$

$$f_x = 5y\,x^{(5y-1)}$$

$$F_y = x^{5y}\ln(x)\cdot 5$$

6. **Question Details**

Find the first partial derivatives of the function.

$f(x, y, z) = 5x\sin(y - z)$

$f_x(x, y, z) =$ $5\sin(y-z)$

$f_y(x, y, z) =$

$f_z(x, y, z) =$

9. **Question Details**

Find the first partial derivatives of the function.

and i.)

$$u = \sin(\overbrace{x_1 + 2x_2 + \cdots + nx_n}^{S_n})$$

$$\frac{\partial u}{\partial x_i} =$$

$$\frac{\partial u}{\partial x_2} = \cos(S_n)\,2$$

$$\frac{\partial u}{\partial x_i} = \boxed{\cos(S_n)\,i}$$

From 14.3(2):

5. **Question Details**

Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

$e^{5z} = xyz$

$\dfrac{\partial z}{\partial x} =$

$\dfrac{\partial z}{\partial y} =$

9. **Question Details**

Find the indicated partial derivative.

$$f(x, y, z) = e^{xyz^4}; \quad f_{xyz}$$

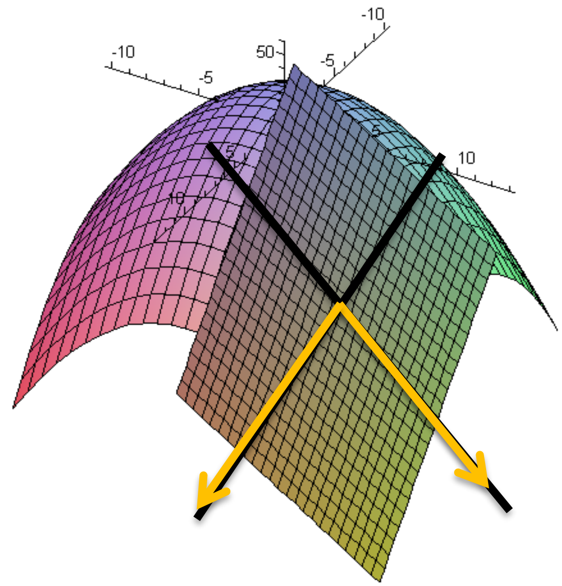
$f_{xyz}(x, y, z) =$

14. **Question Details** SCalcET8 14.3.098. [3800345] -

The paraboloid $z = 8 - x - x^2 - 4y^2$ intersects the plane $x = 1$ in a parabola. Find parametric equations in terms of t for the tangent line to this parabola at the point $(1, 2, -10)$. (Enter your answer as a comma-separated list of equations. Let x, y, and z be in terms of t.)

Hint: See page 7 of these notes for how to get the direction vector in problem 14!!!

14.4 Tangent Planes → 3D Tangent for a surface



The plane that contains all tangent lines to a surface at a point is given by

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

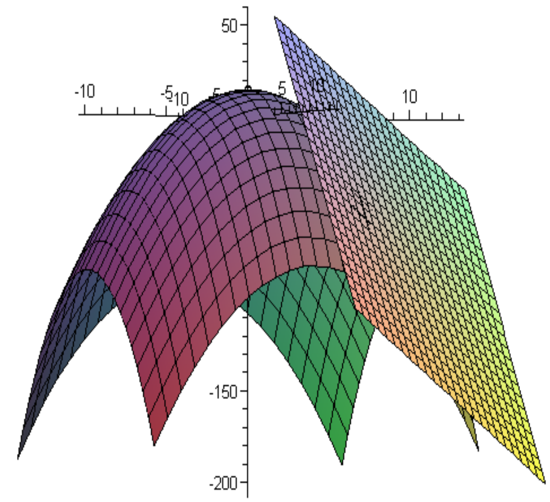
infinatley many

Find 2 tangent lines and find cross product to get normal for plane formula...

General Visuals: https://www.math3d.org/2PoayPs9

Example: Find the tangent plane to

$z = f(x,y) = 15 - x^2 - y^2$ at (7,4)



Step 1: Find…

$f(7,4) = 15 - 49 - 16 = -50 \Rightarrow z_0$

$f_x(7,4) = -2x = -2(7) = -14$

$f_y(7,4) = -2y = -2(4) = -8$

any point on plane

Step 2: Fill in…

$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

(x, y, z) are points on the plane

$$\boxed{z + 50 = -14(x-7) - 8(y-4)}$$

$$\underbrace{z = -50 - 14(x-7) - 8(y-4)}_{\text{linear approximation}}$$

$L(x,y)$

Derivation of Tangent Plane

The plane goes thru (7, 4, -50).
Now we need a normal vector.

y isnt changing

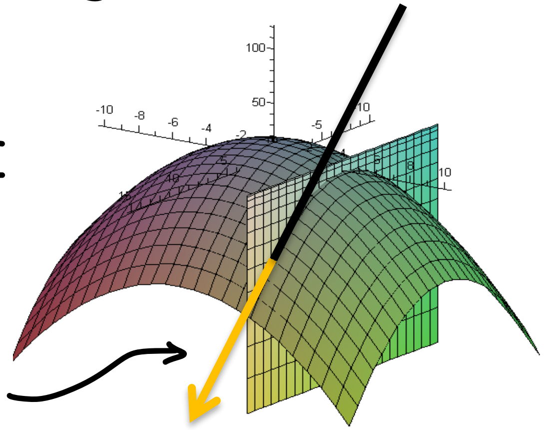
Note:

$f_x(x,y) = -2x$

$f_x(7,4) = -14 = \frac{\Delta z}{\Delta x}$

$\begin{array}{ccc} 1 & 0 & -14 \end{array}$

$\langle \Delta x, \Delta y, \Delta z \rangle$

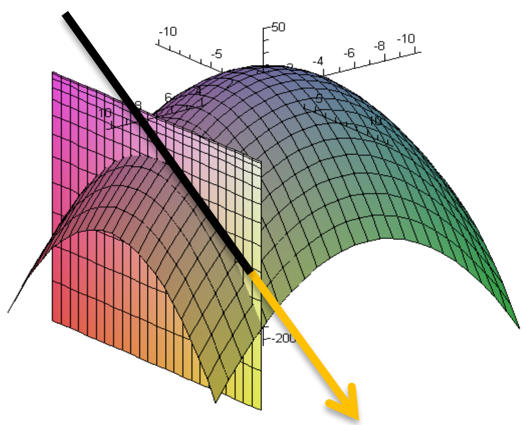


$f_y(x,y) = -2y$

$f_y(7,4) = -8 = \frac{\Delta z}{\Delta y}$

$\begin{array}{ccc} 0 & 1 & -8 \end{array}$

$\langle \Delta x, \Delta y, \Delta z \rangle$



choose numbers that work

Thus, we can get two vectors that are parallel to the plane:

$\langle 1,0,f_x(x_0,y_0)\rangle = \langle 1,0,-14\rangle$

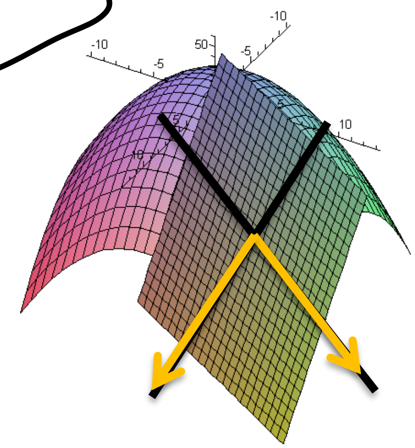
$\langle 0,1,f_y(x_0,y_0)\rangle = \langle 0,1,-8\rangle$

cross product

So a normal vector is given by

$\langle 1,0,-14 \rangle \times \langle 0,1,-8\rangle = \langle \underline{14, 8, 1} \rangle$

plug in



Tangent Plane:

$14(x-7) + 8(y-4) + (z+50) = 0$

Which we rewrite as:

$z + 50 = -14(x-7) - 8(y-4)$

General Derivation

In general, for z = f(x,y) at (x_0, y_0) by:

1. $z_0 = f(x_0, y_0)$ = height.

2. $\langle 1,0, f_x(x_0, y_0)\rangle$ = 'a tangent in *x*-dir.' *(direction vector for tangent line!)*
   $\langle 0,1, f_y(x_0, y_0)\rangle$ = 'a tangent in *y*-dir.'

3. Normal to surface:
   $\langle 1,0, f_x(x_0, y_0)\rangle \times \langle 0,1, f_y(x_0, y_0)\rangle$
   $= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$ $\Rightarrow$ don't worry about this

Tangent Plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$ ☆

An Application of the Tangent Planes

Linear Approximation

"Near" the point (x_0,y_0) the tangent plane and surface z-values are close.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which is the same as

$$L(x, y) = z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Idea:

$$f(x, y) \approx L(x, y) \text{ for } (x, y) \approx (x_0, y_0)$$

F'18 – Exam 2 – Loveless

Find both 1st partial derivatives of

$$f(x,y) = x^5 \sin\left(\frac{\pi x}{y^2}\right) + \ln(y) + 4$$

and give the tangent plane at (1,1).

Use this linear approximation to estimate the value of f(0.95, 1.1).

$$\frac{dz}{dx} = x^5 \cos\left(\frac{\pi x}{y^2}\right)\frac{\pi}{y^2} + 5x^4 \sin\left(\frac{\pi x}{y^2}\right)$$

$$\left.\frac{dz}{dx}\right|_{(1,1)} = \cos(\pi)\pi + 5\sin(\pi) = -\pi$$

$$\frac{dz}{dy} = x^5 \cos\left(\frac{\pi x}{y^2}\right)\left(\frac{-2\pi x}{y^3}\right) + \frac{1}{y}$$

$$\left.\frac{dz}{dy}\right|_{(1,1)} = \cos(\pi)(-2\pi) + 1 = 2\pi + 1$$

$$\boxed{z - 4 = -\pi(x-1) + (2\pi+1)(y-1)}$$

$$z = f(1,1) = 4$$

Visual: https://www.math3d.org/PFEHKAcZ

Sp'17 – Exam 2 – Loveless

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point (2,1,0) for the surface implicitly defined by

$$e^{3z} = x^2 z + \ln(y) + 5x - 10.$$

And give the tangent plane at this point.

Now, consider the curve of intersection of this surface with the fixed plane y = 1, find equations for the tangent line to this curve at (2,1,0).

Visual: https://www.math3d.org/vDdvOSUZ

Sp'18 – Exam 2

Find all 1st and 2nd partials for

$$f(x, y) = x^3 - x^2y + y^2 - 2y.$$

Preview of 14.7: Can you find all (x,y) when BOTH 1st partials equal zero?

Visuals: https://www.math3d.org/ySr7jcXQ